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I demonstrate that instability in a system of homogeneous scalar fields leads to the growth of super-Hubble metric perturbations. This generalizes the result that parametric resonance can lead to the growth of cosmological perturbations. Since dynamical chaos is common in multi-field quartically coupled systems, I argue that the evolution of the fields after inflation must be examined to determine whether the amplitude of cosmological metric perturbations is underestimated in the standard inflationary calculations. I illustrate this effect with a simple hybrid inflation model.

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1. Introduction.— A period of inflation has become the best candidate for the origin of structure in the universe. Recent years have seen considerable interest in the process of reheating, which follows inflation and transforms the inflaton field's large potential energy into particle excitations of various fields. A key development has been the understanding that parametric resonance can lead to extremely rapid particle production in regions of k-space, a process known as preheating [1]. If the preheating dynamics involves more than one scalar field, the resonance can extend to $k = 0$, so that field perturbations which are super-Hubble during preheating are amplified [2,3]. Large scale metric curvature perturbations can be amplified during preheating as well, which contradicts the usual assumption that they are constant on super-Hubble scales. The growth of super-Hubble metric perturbations is possible only when entropy modes are present — single field perturbations are purely adiabatic and hence the usual conservation law applies [4,5]. These large scales are relevant to structure formation and the Cosmic Microwave Background (CMB), and hence it is crucial to understand this stage of the universe's evolution.

In this Letter, I generalize previous studies of parametric resonance in an attempt to understand under what conditions the growth of super-Hubble metric perturbations is possible. I point out a previously undiscussed general route for the amplification of super-Hubble scalar field and metric curvature perturbations in multi-field models, namely through dynamical chaos in the background field evolution. (Note that "dynamical chaos" here refers to the *evolution* of a low-degree-of-freedom dynamical system of homogeneous fields, and should not be confused with "chaotic inflation", which refers to the insensitivity of certain inflationary models to the inflaton *initial conditions*.) The idea is simple to state: if the homogeneous background field dynamics is chaotic, then by definition small homogeneous perturbations of the backgrounds will grow exponentially on some characteristic time scale. This implies the growth of large scale metric perturbations, which are the modes relevant to structure formation and the CMB. Since dynamical chaos is common in nonlinear systems with two or more degrees of freedom (e.g. in quartically coupled oscillators), I stress that the chaotic overproduction of super-Hubble modes may rule out certain inflationary models.

It is important to point out the distinction between parametric resonance and dynamical chaos. While both can result in exponential production of super-Hubble modes, parametric resonance can be described analytically while chaos cannot. Parametric resonance techniques can be applied in the special case that the backgrounds (or conformally scaled backgrounds) undergo periodic motion. Essentially, linear stability analysis is performed about the periodic orbit, and the presence of instability implies exponential growth of perturbations. Dynamical chaos, on the other hand, involves instability to perturbations during extremely complex (and more general) evolution of the backgrounds. The lack of periodic motion (and hence of parametric resonance) does *not* imply that perturbations cannot grow exponentially.

There have been a number of studies of chaos in systems of homogeneous fields in cosmology, although apparently none have made the connection with the growth of super-Hubble perturbations. Easther and Maeda [6] studied the chaotic dynamics of a two-field hybrid inflation system during reheating, although they did not include metric perturbations. They found two effects: the enhancement of defect formation, and a significant variation in the growth of the scale factor. However, they claim that only scales $k \sim aH$ at the time of reheating will be affected. Cornish and Levin [7] studied a single field model, and followed the evolution for several "cosmic cycles" of bang and crunch. Chaos is possible in such a simple system if gravity is important, as we will see below. Latora and Bazeia [8] studied a class of two-field quartically coupled systems which are chaotic in some regions of parameter space.

2. Perturbation dynamics.— A general dynamical system consists of a set of n (in general nonlinear) coupled first order ODEs

$$\dot{x_i} = F_i(x_j). \qquad (1)$$

If we consider the state x_i to be a function of the initial state x_i^0 (and time), we can write

$$\delta x_i = \frac{\partial x_i(x_j^0)}{\partial x_j^0}\delta x_j^0. \qquad (2)$$

In words, Eq. (2) says that once we have obtained a "background solution" $x_i(x_j^0)$ to the equations of motion, we can obtain from it the evolution of a small perturbation simply by taking derivatives. (Of course in specific cases, it may be impossible to obtain the background solution in analytic form in order to implement this method.)

The system (1) is said to exhibit dynamical chaos if the phase space is bounded and if a perturbation length $d(t) = (\delta x^i \delta x_i)^{1/2}$ grows exponentially with time, i.e.

$$d(t) \sim d(t_0)e^{ht}, \qquad (3)$$

for sufficiently small initial displacement $d(t_0)$ and sufficiently late t. Here h is known as the (largest) *Lyapunov exponent* [9]. The requirement of a bounded phase space excludes trivially unstable systems, such as the inverted harmonic oscillator. Necessary conditions for dynamical chaos in the system (1) are that it contain nonlinear terms and that $n \geq 3$ [9], so that even very low dimensional systems can exhibit amplification of perturbations and the consequent extremely erratic background evolution.

I now consider the case of a spatially flat Friedmann-Robertson-Walker background universe with N minimally coupled real scalar fields ϕ_i described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}\sum_i \partial_\mu\phi_i\partial^\mu\phi_i - V(\phi_i)\right). \qquad (4)$$

The equations of motion for the homogeneous background scalar fields are

$$\ddot{\phi}_i + 3H\dot{\phi}_i + V_{,i} = 0, \qquad (5)$$

where $V_{,i} = \partial V/\partial\phi_i$. The background metric evolution is specified by the 0-0 Einstein equation,

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2}\left[\frac{1}{2}\sum_i \dot{\phi}_i^2 + V(\phi_i)\right]. \qquad (6)$$

Note that on the assumption that H does not change sign (which holds in realistic models where $H > 0$), we can substitute the square root of Eq. (6) into Eq. (5). The evolution equations for the set of functions $\{\phi_i, \dot{\phi}_i\}$ can then be written in the form of the dynamical system (1), with $n = 2N$. Thus with two or more nonlinearly coupled scalar fields, dynamical chaos is possible in the homogeneous background system. (Indeed we can see that if we allow H to change sign, as in bang-crunch scenarios, then we cannot eliminate H from Eq. (5). In this case $n = 2N+1$, and a single scalar field is sufficient for chaos, as was found in [7].)

We can write the metric in the presence of scalar perturbations in the general form [10]

$$ds^2 = (1+2A)dt^2 - a^2(t)\{2B_{|i}dx^i dt - [(1-2\psi)\delta_{ij} + 2E_{|ij}]dx^i dx^j\}. \qquad (7)$$

Here A, B, ψ, and E are scalar functions, and subscript $|i$ refers to the covariant derivative on the background constant time hypersurface. Two of the four scalar functions may be determined by a choice of gauge, and the linear perturbation evolution equations will reflect that choice. To obtain the long-wavelength limit of these equations, Sasaki and Tanaka [11] (see also [12]) noticed that it was sufficient to use Eq. (2) (in a slightly different form). The issue of which gauge the perturbation equations are written in is tied to the choice of time variable used in the background equations [11]. Explicitly, they found that to obtain the perturbed equations in a particular gauge, it is necessary to use a time variable which is *not perturbed* in that gauge.

For example, in the synchronous gauge, $A = B = 0$, the usual cosmological time t itself remains unperturbed. Thus Eq. (2) applied to the background equations (5) gives

$$\delta\ddot{\phi}_i + 3\delta H\dot{\phi}_i + 3H\delta\dot{\phi} + V_{,ij}\delta\phi_j = 0. \qquad (8)$$

With the identification

$$\delta H = \dot{\psi} \qquad (9)$$

we obtain the usual perturbation equation of motion in the synchronous gauge, in the limit $k \to 0$.

The importance of this result is that, if the homogeneous background dynamics is chaotic, the quantity $\delta\phi_i$ calculated above (which represents the homogeneous field perturbation in a particular gauge) *will* generically grow exponentially with time, as Eq. (3) indicates. Since spatial derivative terms are negligible on super-Hubble scales, this implies the exponential growth of super-Hubble finite wavelength modes. Finally, we can write the comoving curvature perturbation $\mathcal{R}$ by [5]

$$\mathcal{R} = \psi + \frac{H}{\dot{\sigma}}\delta\sigma, \qquad (10)$$

where σ and $\delta\sigma$ are the field and perturbation components along the direction of the background trajectory. Since the total length of the perturbation vector grows exponentially (unless the initial perturbation happens to be precisely adiabatic, corresponding to a perturbation in time), so must the component $\delta\sigma$, and hence so generically must $\mathcal{R}$.

3. Hybrid inflation.— I can illustrate these ideas with a double scalar field model, which, as discussed above, is sufficiently complex for dynamical chaos to be possible. A popular class of two-field inflationary models is *hybrid* inflation [13,14]. In these models inflation can be terminated by a symmetry breaking transition in one of the fields, and the subsequent oscillations can be chaotic [6]. I will consider the potential

$$V(\phi,\chi) = \frac{1}{4\lambda}(M^2 - \lambda\chi^2)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\chi^2. \quad (11)$$

Inflation occurs at large ϕ, where the effective mass of the χ field, $m_\chi^2 = g^2\phi^2 + \lambda\chi^2 - M^2$, is large and the χ field sits at the bottom of a potential valley at $\chi = 0$. Once the inflaton field drops below the critical value $\phi_c = M/g$, the mass-squared m_χ^2 becomes negative (the potential valley becomes a ridge), and the fields undergo a symmetry breaking transition to one of the global minima at $\phi = 0$, $\chi = \pm\chi_0$, where $\chi_0 = M/\sqrt{\lambda}$. I consider the "vacuum-dominated" regime, where the potential during the inflationary stage, $V(\phi) = M^4/(4\lambda) + m^2\phi^2/2$, is dominated by the false vacuum energy term $M^4/(4\lambda)$. I also consider the case where the Hubble parameter at the critical point,

$$H_0^2 = \frac{2\pi}{3\lambda}\frac{M^4}{m_{\rm Pl}^2}, \quad (12)$$

is much smaller than the oscillation frequencies about the global minima, which are $\overline{m}_\phi = gM/\sqrt{\lambda}$ and $\overline{m}_\chi = \sqrt{2}M$ for small oscillations. This ensures that the fields will oscillate very many times after the critical point is reached before Hubble damping is significant.

Preheating has been studied in hybrid models for various parameter regimes in the absence of metric perturbations [15]. The behaviour of large-scale metric perturbations was studied in [16], where it was found that growth is possible on large scales. Oscillations in hybrid inflation were found to be chaotic in [6], although for a very different parameter regime than I examine here. Also, the connection with the growth of large-scale perturbations was not made in [6].

Since my interest in this Letter is to establish a kinematical connection between dynamical chaos in the background fields and exponential growth of metric perturbations, I ignored the evolution of perturbations during the inflationary stage. It is important to note that for the parameters I consider, the large χ mass during inflation implies damping of large scale perturbations during inflation. Thus the question of whether the amplitude of metric perturbations produced in this model is consistent with the Cosmic Background Explorer normalization is not addressed here. A careful analysis, following the evolution of all important scales during inflation and preheating, and including the effects of backreaction, is required [3].

I considered the slice through parameter space specified by $M = 10^{-8}m_{\rm Pl}$, $m = 10^{-16}m_{\rm Pl}$, $\lambda = 10^{-3}$, and $g^2 = 10^{-2}$–10^{-4}. These parameters give an amplitude of cosmological density perturbations of the order 10^{-5} according to the standard inflationary calculation [14]. I evolved the homogeneous background fields according to Eqs. (5) with initial conditions $\phi(t_0) = 0.999\phi_c$ and $\chi(t_0) = 0.001\chi_0$. I followed the evolution of the curvature perturbation on uniform-density hypersurfaces ζ_k (which coincides with $\mathcal{R}$ on large scales) for $k/a = 10^{-3}H_0$ using the longitudinal gauge equations

$$\delta\ddot{\phi}_i + 3H\delta\dot{\phi}_i + \frac{k^2}{a^2}\delta\phi_i + V_{,ij}\delta\phi_j = 4\dot{\Phi}\dot{\phi}_i - 2V_{,i}\Phi, \quad (13)$$

$$\dot{\Phi} + H\Phi = \frac{4\pi}{m_{\rm Pl}^2}\dot{\phi}_i\delta\phi_i, \quad (14)$$

$$\zeta_k = \Phi_k - \frac{H}{\dot{H}}\left(\dot{\Phi}_k + H\Phi_k\right), \quad (15)$$

where Φ_k is the longitudinal gauge metric perturbation. I calculated the largest Lyapunov exponent h for the background evolution using Eq. (3). The results, shown in Fig. 1, indicate the presence of rich structure as g^2 is varied. Regions of chaos with $h \simeq 0.05M$ are interspersed with regular stability bands where $h \simeq 0$. The most prominent stability band is near the supersymmetric point $g^2/\lambda = 2$. The correlation between the Lyapunov exponent and the growth rate of large scale metric perturbations is strong numerical evidence in support of my arguments.

The growth at large scales is not simply due to the negative mass-squared (or "tachyonic") instability near the potential ridge at $\chi = 0$ [17]. To demonstrate this, I plot in Fig. 2 the logarithmic growth rate of $\delta\chi_k$ perturbations as a function of scale k. The solid line corresponds to the same parameters as in Fig. 1 (with $g^2 = 10^{-3}$), and shows a growth rate which is large at scales $k \simeq aM$ (due to the tachyonic instability), and approaches a constant (the chaotic Lyapunov exponent) as $k/(aH) \to 0$. The dotted line shows the results for the same parameters except with the field trajectory constrained to $\phi = 0$. In this case, there is no large-scale growth (and no chaos) as expected for the effectively one-dimensional dynamics, while there is strong growth (due to the tachyonic instability) at small scales. Thus the two effects are distinct, since both trajectories pass through the tachyonic instability region, while only one exhibits growth on large scales.

4. Summary.— I have demonstrated a general link between instability in scalar field background evolution and the growth of super-Hubble metric perturbations. In particular, dynamical chaos in the fields can drive the growth — it is not necessary to have periodic motion and

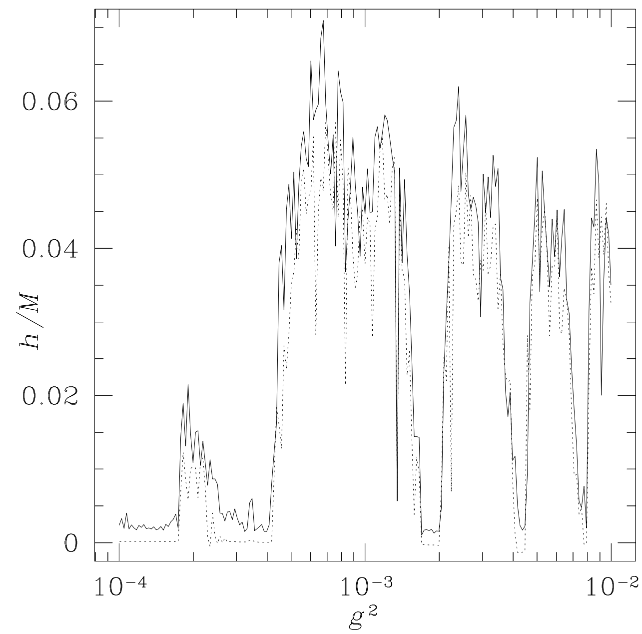

FIG. 1. Largest Lyapunov exponent (solid line) for the homogeneous fields, and logarithmic growth rate of ζ_k (dotted line) for a scale $k/a = 10^{-3}H_0$. The homogeneous fields are oscillating about one of the hybrid model's global minima. The parameters are $M = 10^{-8}m_{\rm Pl}$, $m = 10^{-16}m_{\rm Pl}$, and $\lambda = 10^{-3}$. There is a clear correlation between the two curves.

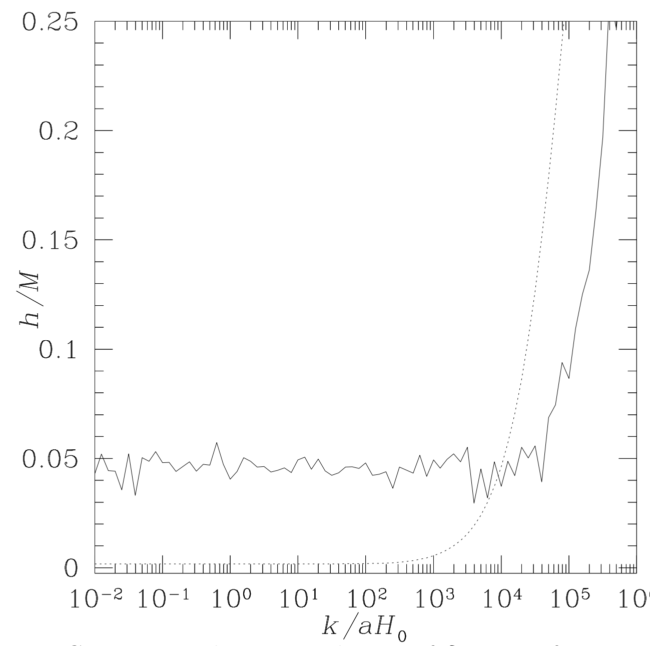

FIG. 2. Logarithmic growth rate of $\delta\chi_k$ as a function of scale for the parameters of Fig. 1 and $g^2 = 10^{-3}$ (solid line) and for the trajectory constrained to $\phi = 0$ (dotted line).

parametric resonance in the fields. Since chaos is common in multi-field systems, it is important to examine the super-Hubble evolution during preheating carefully, and also to follow the evolution during the inflationary period, in order to determine whether the model conflicts with CMB measurements.

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